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Probability distribution of the index in gauge theory on 2d non-commutative geometry

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Abstract: We investigate the effects of non-commutative geometry on the topological aspects of gauge theory using a non-perturbative formulation based on the twisted reduced model. The configuration space is decomposed into topological sectors labeled by the index ν of the overlap Dirac operator satisfying the Ginsparg-Wilson relation. We study the probability distribution of ν by Monte Carlo simulation of the U(1) gauge theory on 2d non-commutative space with periodic boundary conditions. In general the distribution is asymmetric under $\nu \mapsto -\nu$, reflecting the parity violation due to non-commutative geometry. In the continuum and infinite-volume limits, however, the distribution turns out to be dominated by the topologically trivial sector. This conclusion is consistent with the instanton calculus in the continuum theory. However, it is in striking contrast to the known results in the commutative case obtained from lattice simulation, where the distribution is Gaussian in a finite volume, but the width diverges in the infinite-volume limit. We also calculate the average action in each topological sector, and provide deeper understanding of the observed phenomenon.

Keywords: Nonperturbative Effects, Solitons Monopoles and Instantons, Non-Commutative Geometry.

Contents

1. Introduction

Non-commutative (NC) geometry [1, 2] has been studied for quite a long time as a simple modification of our notion of space-time at short distances possibly due to effects of quantum gravity [3]. It has attracted much attention since it was shown to appear naturally from matrix models [4, 5] and string theories [6]. In particular, field theory on NC geometry has a peculiar property known as the UV/IR mixing [7], which may cause a drastic change of the long-distance physics through quantum effects. This phenomenon has been first discovered in perturbation theory, but it was shown to appear also in a fully nonperturbative setup [8]. A typical example is the spontaneous breaking of the translational symmetry in NC scalar field theory, which was first conjectured from a self-consistent one-loop analysis [9] and confirmed later on by Monte Carlo simulation [10 − 12]. (See also [13, 14].)

The appearance of a new type of IR divergence due to the UV/IR mixing spoils the perturbative renormalizability in most cases [15], and therefore, even the existence of a sensible field theory on a NC geometry is a priori debatable. In order to study such a nonperturbative issue, one has to define a regularized field theory on NC geometry. This can be done by using matrix models. In the case of NC torus, for instance, the so-called twisted reduced model [16, 17] is interpreted as a lattice formulation of NC field theories [8], in which finite N matrices are mapped one-to-one onto fields on a periodic lattice. The existence of a sensible continuum limit and hence the nonperturbative renormalizability have been shown by Monte Carlo simulations in NC U(1) gauge theory in 2d [18] and 4d [19] as well as in NC scalar field theory in 3d [12, 20].

In the case of fuzzy sphere [21], finite N matrices are mapped one-to-one onto functions on the sphere with a specific cutoff on the angular momentum. The fuzzy sphere (or fuzzy manifolds [22, 23] in general) preserves the continuous symmetry of the base manifold, which makes it an interesting candidate for a novel regularization of *commutative* field theories alternative to the lattice [24]. Stability of fuzzy manifolds in matrix models with the Chern-Simons term [25, 26] has been studied by Monte Carlo simulations [27, 28].

One of the interesting features of NC field theories is the appearance of a new type of topological objects, which are referred to as NC solitons [29], NC monopoles, NC instantons, and fluxons [30] in the literature. They are constructed by using a projection operator, and the matrices describing such configurations are assumed to be infinite dimensional. In finite NC geometries topological objects have been constructed by using the algebraic K-theory and projective modules [31 – 33].

Dynamical aspects of these topological objects are of particular importance in the realization of a chiral gauge theory in our four-dimensional world by compactifying a string theory with a nontrivial index in the compactified dimensions. Ultimately we hope to realize such a scenario *dynamically*, for instance, in the IIB matrix model [34], in which the dynamical generation of *four-dimensional* space-time [35 – 37] as well as the gauge group [38, 39] has been studied intensively. A crucial link in generating chiral fermions from a matrix model is provided by the index theorem [40], which relates the topological charge of an arbitrary gauge configuration to the index of the Dirac operator on that background. The index theorem can be proved in noncommutative $\mathbb{R}^d$ in the same way as in the commutative case [41].

Extension of the index theorem to *finite* NC geometry is a non-trivial issue due to the doubling problem of the naive Dirac action. In lattice gauge theory, an analogous problem was solved by the use of the so-called overlap Dirac operator [42 – 45], which satisfies the Ginsparg-Wilson relation [46]. The ideas developed in lattice gauge theory have been successfully extended to NC geometry. In the case of NC torus, the overlap Dirac operator has been introduced in ref. [47], and it was used to define a NC chiral gauge theory with manifest star-gauge invariance. For general NC manifolds, a prescription to define the Ginsparg-Wilson Dirac operator and its index has been provided in ref. [48], and the fuzzy sphere was considered as a concrete example.[1] The Ginsparg-Wilson algebra for the fuzzy sphere has been studied in detail in each topological sector [32]. In ref. [50] the overlap Dirac operator on the NC torus [47] was derived also from this general prescription [48], and the axial anomaly has been calculated in the continuum limit.

In an attempt to construct a topologically nontrivial configuration on the fuzzy sphere, an analogue of the 't Hooft-Polyakov monopole was obtained [32, 33]. Although the index defined through the Ginsparg-Wilson Dirac operator vanishes for these configurations, one can make it non-zero by inserting a projection operator, which picks up the unbroken U(1) component of the SU(2) gauge group. In fact the 't Hooft-Polyakov monopole configurations are precisely the meta-stable states observed in Monte Carlo simulations [27] taking the two coincident fuzzy spheres as the initial configuration, which eventually decays into a single fuzzy sphere. In ref. [51] this instability was studied analytically by the one-loop calculation of free energy around the 't Hooft-Polyakov monopole configurations, and it was interpreted as the dynamical generation of a nontrivial index, which may be used for the realization of a chiral fermion in our space-time.

In our previous work [52], we have demonstrated the validity of the index theorem in finite NC geometry, taking the 2d U(1) gauge theory on a discretized NC torus as a

[1] The Ginsparg-Wilson Dirac operator for vanishing gauge field was constructed earlier in refs. [49].

simple example, which is studied extensively in the literature both numerically [18] and analytically [53 – 55]. In particular, ref. [55] presents general classical solutions carrying the topological charge. We computed the index defined through the Ginsparg-Wilson Dirac operator for these classical solutions and compared the results with the topological charge. The index theorem holds when the action is small, but the index takes only multiple integer values of N, the size of the 2d lattice. For non-zero indices, the action is finite in the large N limit, but it diverges when the bare coupling constant is tuned in the continuum limit. By interpolating the classical solutions, we constructed explicit configurations for which the index is of order 1, but the action becomes of order N. These results suggested that the probability of obtaining a non-zero index vanishes in the continuum limit.

In this paper we confirm this statement *at the quantum level* by performing Monte Carlo simulation of the 2d U(1) gauge theory on a NC discretized torus. Since the theory is known to have a sensible continuum limit [18], we investigate the probability distribution of the index in that limit. Comparison with the known results in the corresponding commutative case obtained from lattice simulation [56] allows us to reveal the striking effects of NC geometry.

The rest of this paper is organized as follows. In section 2 we define the model and the index of the overlap Dirac operator. In section 3 we show our results for the probability distribution of the index. In section 4 we discuss the average action in each topological sector, which provides qualitative understanding for the behavior of the probability distribution. Section 5 is devoted to a summary and discussions.

2. The model and the topological sectors

In this section we define the model and the topological sectors based on the matrix model formulation of NC gauge theory. For more details such as the interpretation of matrices as fields on a NC torus, we refer the reader to our previous paper [52].

The model we study in this paper is given by the action

$$S = N^2\beta \sum_{\mu\neq\nu} \left\{ 1 - \frac{1}{N} \mathcal{Z}_{\nu\mu} \, \mathrm{tr} \left(V_\mu \, V_\nu \, V_\mu^\dagger \, V_\nu^\dagger \right) \right\} , \tag{2.1}$$

where $\mathcal{Z}_{\mu\nu} = \mathcal{Z}_{\nu\mu}^*$ is a phase factor given by [47]

$$\mathcal{Z}_{12} = \exp \left(\pi i \frac{N+1}{N} \right) \tag{2.2}$$

with N being an odd integer. The NC tensor $\Theta_{\mu\nu}$, which characterizes NC geometry $[x_\mu, x_\nu] = i\Theta_{\mu\nu}$, is given by

$$\Theta_{\mu\nu} = \vartheta \, \epsilon_{\mu\nu} \, , \qquad \vartheta = \frac{1}{\pi} N a^2 \, . \tag{2.3}$$

Since the NC parameter ϑ is related to the lattice spacing by (2.3), we have to take the large N limit together with the continuum limit $a \to 0$ in order to obtain a continuum theory with finite ϑ. In that limit the physical extent of the torus $\ell = aN$ goes to infinity

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at the same time. Whether one can obtain a sensible continuum limit by tuning β appropriately is a non-trivial issue, which has been addressed in ref. [18]. It turned out that β should be sent to ∞ as

$$\beta \propto \frac{1}{a^2} \; . \tag{2.4}$$

Combining this with (2.3), one finds that the large N limit should be taken together with $\beta \to \infty$ limit so that β/N is fixed. This limit is called the "double scaling limit", in which non-planar diagrams survive. If one takes the planar limit ($N \to \infty$ with fixed β) instead, one obtains a gauge theory on a NC space with $\vartheta = \infty$. In this limit the Wilson loops agree [2] with the SU(∞) lattice gauge theory [57] due to the Eguchi-Kawai equivalence [16]. In particular the expectation value of the action in this limit is given by [57]

$$\langle S \rangle = \begin{cases} 2\beta N^2 (1-\beta) & \text{for } \beta < \frac{1}{2} \, , \\ \frac{1}{2} N^2 & \text{for } \beta \geq \frac{1}{2} \, , \end{cases} \tag{2.5}$$

which shows that the system undergoes a third order phase transition at $\beta = \beta_{\rm cr} \equiv 1/2$.

The configuration space can be naturally decomposed into topological sectors by the index of the overlap Dirac operator on the discretized NC torus [47, 48, 50]. Let us define the covariant forward (backward) difference operator

$$\begin{aligned} \nabla_\mu \Psi &= \frac{1}{a} \left[V_\mu \Psi \Gamma_\mu - \Psi \right] \, , \\ \nabla_\mu^* \Psi &= \frac{1}{a} \left[\Psi - V_\mu^\dagger \Psi \Gamma_\mu \right] \, , \end{aligned} \tag{2.6}$$

where the SU(N) matrices Γ_μ ($\mu = 1, 2$) satisfy the 't Hooft-Weyl algebra

$$\Gamma_\mu \Gamma_\nu = \mathcal{Z}_{\mu\nu} \Gamma_\nu \Gamma_\mu \; . \tag{2.7}$$

Given the covariant forward (backward) difference operator, we can define the overlap Dirac operator in precisely the same way as in the commutative case.

First the Wilson-Dirac operator can be defined as

$$D_{\rm W} = \frac{1}{2} \sum_{\mu=1}^{2} \left\{ \gamma_\mu \left(\nabla_\mu^* + \nabla_\mu \right) - a \nabla_\mu^* \nabla_\mu \right\} \, , \tag{2.8}$$

where γ_μ ($\mu = 1, 2$) are the gamma matrices in 2d. A crucial property of the overlap Dirac operator D is the Ginsparg-Wilson relation [46]

$$\gamma_5 D + D \gamma_5 = a \, D \gamma_5 D \, , \tag{2.9}$$

where $\gamma_5 = -i \gamma_1 \gamma_2$ is the chirality operator. Assuming the γ_5-hermiticity $D^\dagger = \gamma_5 D \gamma_5$, we can define a hermitean operator $\hat{\gamma}_5$ by

$$\hat{\gamma}_5 = \gamma_5 \left(1 - a D \right) \, , \tag{2.10}$$

[2] At finite ϑ, the agreement holds only when the physical area surrounded by the Wilson loop is much smaller than ϑ. As a consequence, the relation (2.4) agrees with the one required for the continuum limit of the SU(∞) lattice gauge theory [57].

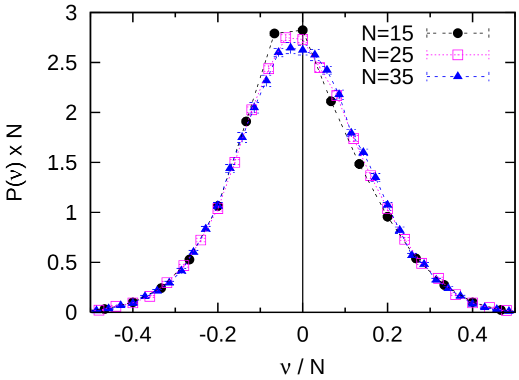

Figure 1: The probability distribution of $\frac{\nu}{N}$ for various N at $\beta = 0$.

which may be solved for D as $D = \frac{1}{a}(1 - \gamma_5\hat{\gamma}_5)$. Then the Ginsparg-Wilson relation (2.9) is equivalent to requiring $\hat{\gamma}_5$ to be unitary. The overlap Dirac operator corresponds to taking $\hat{\gamma}_5$ to be [42]

$$\hat{\gamma}_5 = \frac{H}{\sqrt{H^2}}, \tag{2.11}$$

$$H = \gamma_5 (1 - aD_{\mathrm{W}}) , \tag{2.12}$$

where D_{W} is the Wilson-Dirac operator.

One can define the index of D unambiguously by $\nu \equiv n_+ - n_-$, where $n_\pm$ is the number of zero modes with the chirality ± 1. It turns out that [43–45]

$$\nu = \frac{1}{2}\mathcal{T}r(\gamma_5 + \hat{\gamma}_5) = \frac{1}{2}\mathcal{T}r\frac{H}{\sqrt{H^2}}, \tag{2.13}$$

where $\mathcal{T}r$ represents a trace over the space of matrices and over the spinor index.

We performed Monte Carlo simulation of the model (2.1) using the heat bath algorithm as in ref. [18]. For each configuration V_μ generated by simulation, we calculate the index (2.13). We diagonalize the hermitean matrix H defined by (2.12), and count the number of positive and negative eigenvalues. (Note that the lattice spacing a which appears in the expressions (2.6), (2.8) and (2.12) cancel each other, and the index does not depend explicitly on a.) The computational effort for calculating the index is of order N^6, since we have to diagonalize the $2N^2 \times 2N^2$ hermitean matrix H.

3. Probability distribution of the index

In this section we present our results for the probability distribution of the index ν — as computed by the definition (2.13) — in the gauge theory on the NC torus.

In figure 1 we plot the probability distribution of ν for various N at $\beta = 0$. This represents the distribution in the configuration space without taking account of the action. To our surprise, it turns out that the distribution of ν is asymmetric under $\nu \mapsto -\nu$. This is in striking contrast to ordinary commutative theories, in which the distribution

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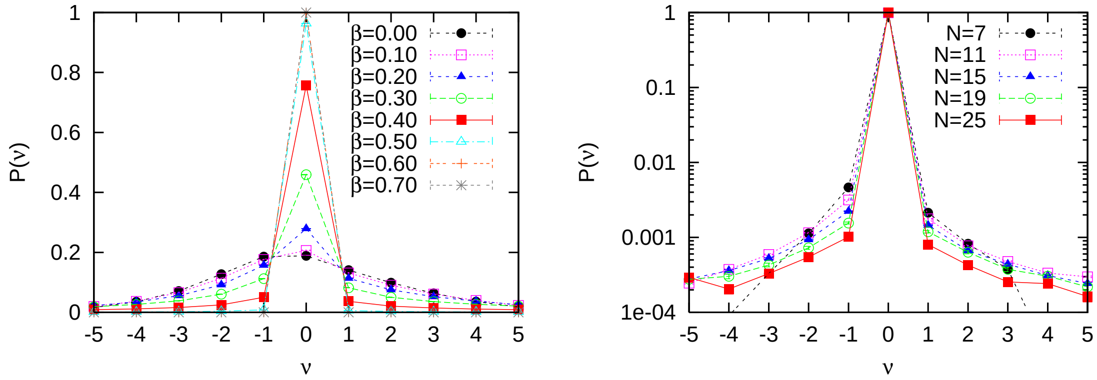

Figure 2: The probability distribution of ν is plotted for various β at $N = 15$ (left) and for various N at $\beta = 0.55$ (right). In the right plot, the log scale is taken for the y-axis to magnify the results at $\nu \neq 0$.

of ν is symmetric due to parity invariance. We also find that the distribution for the rescaled topological charge ν/N at different N lies on top of each other. This behavior is analogous to what one obtains in the commutative continuum theory (See, for instance, section 6 of ref. [52].). The plot also confirms the existence of $\nu \neq 0$ configurations on the discretized NC torus. An example of such configurations is found numerically in ref. [58], and constructed analytically in section 5 of ref. [52]. The crucial question we address in what follows is whether such configurations survive in the continuum limit.

Let us see how the probability distribution of ν changes as we switch on β. In figure 2 (left) we plot the probability distribution $P(\nu)$ for various β at $N = 15$. (Throughout this paper, we assume the normalization $\sum_\nu P(\nu) = 1$.) We find that the probability for $\nu \neq 0$ decreases rapidly, and the probability for $\nu = 0$ approaches unity. In figure 2 (right) we plot the probability distribution $P(\nu)$ for various N at $\beta = 0.55$. Note that the value of β we have chosen is above the critical point $\beta = \beta_{\rm cr} \equiv 1/2$ of the Gross-Witten phase transition. We find that the distribution approaches the Kronecker delta $\delta_{\nu 0}$ not only for increasing β but also for increasing N. In figure 3 we plot the ratio $P(\nu)/P(0)$ for $\nu = 1, -1$ for various β at $N = 15$ (left) and for various N at $\beta = 0.55$ (right). In both cases we observe an exponentially decreasing behavior.

As we mentioned in the previous section, in order to take the continuum limit, we have to send N and β to ∞ simultaneously fixing the ratio β/N. It is clear from the above results that the distribution $P(\nu)$ approaches $\delta_{\nu 0}$ very rapidly in that limit.

4. Average action in each topological sector

In this section we provide an explanation of our results in the previous section by studying the action in each topological sector. In figure 4 we plot the distribution of the action S for $\nu = 0, -1$ at $\beta = 0.1$ and $\beta = 0.5$. We find that at $\beta = 0.1$ the distribution for different topological sector lies on top of each other, while at $\beta = 0.5$ the distribution for $\nu = 0$

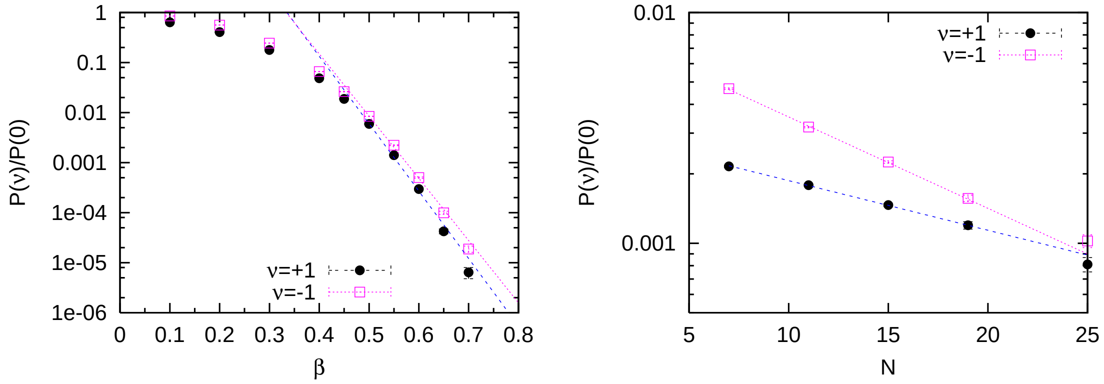

Figure 3: The ratio $P(\nu)/P(0)$ for $\nu = 1, -1$ is plotted in the log scale as a function of β at $N = 15$ (left) and as a function of N at $\beta = 0.55$ (right). The straight lines represent a fit to an exponentially decreasing behavior.

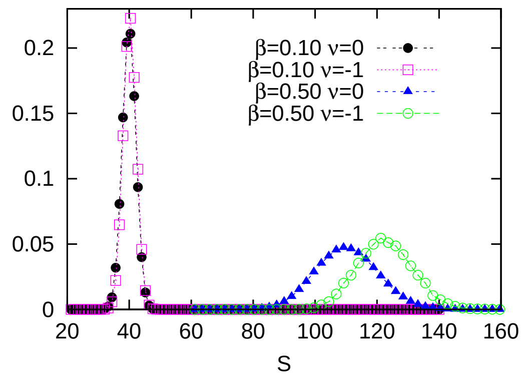

Figure 4: The distribution of the action in the $\nu = 0$ and $\nu = -1$ topological sectors is plotted for $\beta = 0.1$ and $\beta = 0.5$ at $N = 15$.

differs much from $\nu = -1$. We have also measured the distribution for $\nu = 1, 2, -2$, which turns out to be very close to the result for $\nu = -1$.

In figure 5 we plot the average value of the action $\bar{S}(\nu)$ in each topological sector. We find that the result is almost flat except at $\nu = 0$. Note that the weighted sum of $\bar{S}(\nu)$ yields

$$\sum_{\nu} \bar{S}(\nu) P(\nu) = \langle S \rangle \,, \tag{4.1}$$

where $\langle S \rangle$ is given by (2.5) in the planar limit. When the $\nu = 0$ sector dominates, we have $\bar{S}(0) \simeq \langle S \rangle$. This explains the behavior of $\bar{S}(0)$ in figure 5.

In both plots in figure 5, we observe a dip at $\nu = 0$. In figure 6 we plot the size of the dip defined by

$$\Delta S \equiv \bar{S}(-1) - \bar{S}(0) \,, \tag{4.2}$$

which shows that the dip grows linearly with both β and N. (From the left plot, we find that the linear behavior sets in at $\beta \sim 0.5$, which is close to the critical point of the Gross-

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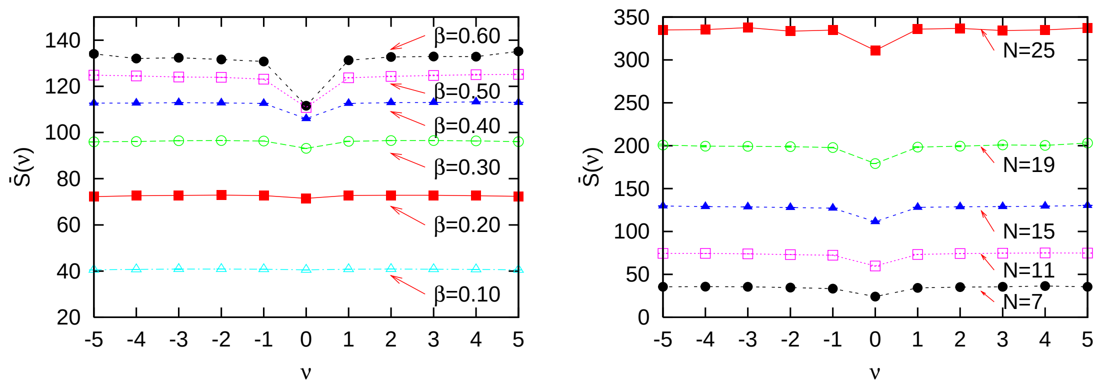

Figure 5: The average value of the action is plotted against the index ν for various β at $N = 15$ (left) and for various N at $\beta = 0.55$ (right).

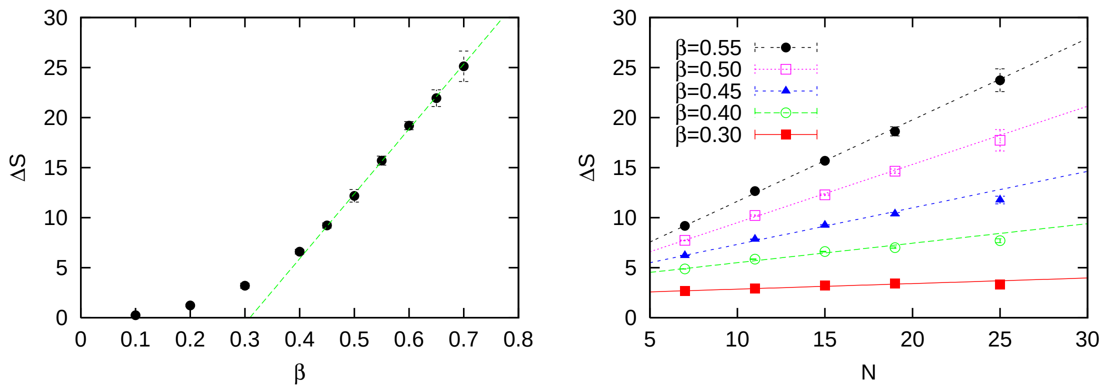

Figure 6: The dip ΔS is plotted as a function of β for $N = 15$ (left) and as a function of N for various β (right).

Witten phase transition.) This is consistent with the exponentially decreasing behavior of the probability $P(\nu)/P(0)$ for $\nu \neq 0$ discussed in the previous section.

In the commutative case [56], lattice simulation shows that the average action in each sector increases quadratically with ν, but the coefficient vanishes in the infinite-volume limit. Correspondingly the distribution of ν is Gaussian in a finite volume, but the width diverges in the infinite-volume limit. Thus the situation in the NC case differs drastically from the commutative case.

5. Summary and discussions

In this paper we have studied the effects of NC geometry on the probability distribution of the index ν of the Dirac operator. In the 2d U(1) gauge theory with periodic boundary conditions, we found that the probability for $\nu \neq 0$ is exponentially suppressed in the continuum and infinite-volume limits. Our conclusion is consistent with our previous analysis at the classical level [52] and with the instanton calculus in the continuum theory [54]. In fact the topologically trivial sector includes all the instanton configurations that contribute to the partition function.

In order to understand our conclusion intuitively, let us recall that in NC geometry, the action involves the star product, which must have certain smoothening effects on the gauge field. In the commutative case with periodic boundary conditions, a classical solution in a topologically non-trivial sector has a constant field strength, but the vector potential has a singularity. (See e.g., section 6 of ref. [52].) It is therefore conceivable that such configurations cannot be realized in NC geometry. Our results in section 4 substantiate this argument.

It follows from our conclusion that special care must be taken when one studies the θ-vacuum in NC geometry.[3] In general one has to sum over (topologically different) twisted boundary conditions labeled by ν with the phase factor $e^{i\nu\theta}$. In the commutative case, however, one may equivalently add a θ-term to the action and just integrate over the lattice configuration with periodic boundary conditions, as is done e.g., in ref. [60]. Our conclusion implies that this is no longer true in NC geometry. In ref. [52] we speculated that NC geometry may provide a solution to the strong CP problem, but this remains to be seen.

As another effect of NC geometry, we found that in general the probability distribution of ν becomes asymmetric under $\nu \mapsto -\nu$, reflecting the parity violation due to NC geometry. This is interesting since it suggests a possibility to obtain a non-zero vacuum expectation value for the index ν in some NC model. Alternatively, one can twist the boundary condition to make a topologically non-trivial sector dominate in the continuum and infinite-volume limits [59]. We expect that these unusual properties of NC geometry may provide a dynamical mechanism for realizing chiral fermions in string theory compactifications, or a mechanism for generating non-zero baryon number density in the universe. See refs. [51, 61] for a related line of research using fuzzy spheres in the extra dimensions.

From the motivations mentioned above, it would be interesting to extend the present analysis to four dimensions. Unlike the 2d case studied in this paper, the perturbative vacuum is actually unstable due to the UV/IR mixing [62–67]. However, the system stabilizes after the condensation of the Wilson lines and finds a stable nonperturbative vacuum [19], in which the translational invariance is spontaneously broken. One can also stabilize the perturbative vacuum by keeping the UV cutoff finite and regarding the model as a low-energy effective theory. The situation may depend on which vacuum one chooses. We hope to address such issues in future publications.

Acknowledgments

It is our pleasure to thank Hidenori Fukaya, Satoshi Iso, Hikaru Kawai, Kenji Ogawa and Kentaroh Yoshida for valuable discussions.

[3] Strictly speaking, we need to have the ordinary (commutative) time in order to be able to speak about a "vacuum". We may think of, for instance, four-dimensional space-time with non-commutativity introduced only in two spatial directions [19]. Let us also remind the readers that the parameter θ should not be confused with the non-commutativity parameter ϑ.

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